# Elvis Lives: Mathematical Surprises Inspired by Elvis, the Welsh Corgi 

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On August 2, 2013, newspapers around the country, including USA Today, carried the news of the death of a dog. Even more remarkable, the dog achieved his fame because he knew calculus.

Elvis, a one-year-old Welsh corgi, came to live with the third author in August 2001. Given his namesake, maybe fame was inevitable. (He got his name because when he nursed, his back legs pumped back and forth.) He was a retriever at heart who, in his enthusiasm to find a stick, once unearthed a young sapling to throw and another time dragged a $\log$ from a fire pit-still smoldering on one end. But he gained his fame by finding the optimal (quickest) route to a tennis ball thrown into Lake Michigan. Starting at the water's edge, Elvis sprinted down the shore and then dove into the water, usually within a foot of the optimal point $P_{o}$ shown in Figure 1. The resulting article, "Do Dogs Know Calculus?" [5], catapulted him into a never-ending series of newspaper articles, radio spots, textbooks and popular mathematics books, and a lecture circuit that approached 200 talks given throughout the U.S. and abroad.

Through it all, Elvis was a tireless ambassador for mathematics. Whether the audience was a classroom of professors and graduate students at Notre Dame or Boston University, bleachers filled with 500 undergraduates at Roanoke College, or a seven-year-old child on the beach, Elvis stole hearts. After the formulas were derived, the

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audience was instructed to discern whether Elvis really knew calculus by watching his eyes and ears as the words were slowly mouthed, "Elvis, what is the derivative of $x$ cubed?" Then after a pregnant pause of eager anticipation, "There, that's what he always does-he doesn't do a thing!"

But the humor revealed a valuable lesson. Elvis was just one example of the myriad ways in which nature finds optimal solutions, naturally. Whether it be soap bubbles forming the minimum surface area, rays of light (like Elvis) finding the quickest path, or leaves finding the best system of vessels, nature is fascinating in its efficiency.

As this article will demonstrate, Elvis's behavior continues to inspire the discovery of a number of surprising mathematical insights. The original article [5] showed the unexpected result that Elvis's entry point to the water is independent of the length of the throw down the beach. In other words, the point of entry toward the ball located at $P_{b}=\left(x_{b}, y_{b}\right)$ is given by

$$
x_{o}=x_{b}-\frac{y_{b}}{\sqrt{\left(\frac{r}{s}\right)^{2}-1}}
$$

where $r$ is the running speed and $s$ the swimming speed; see Figure 1 for an example. Then Perruchet and Gallego [6] explained that their Labrador, Salsa, behaved similarly, and proved the following: If the strategy, instead, is to move at each point along the path toward the ball as quickly as possible, then the resulting path would be identical to the one obtained by seeking a global minimum. In other words, the entry point $P_{o}$ would not change.


Figure 1. An example of Elvis's path with $r=2$ and $s=1$. Starting at $P_{a}=(0,0)$ and ending at $P_{b}=(3,1)$, Elvis runs along the shore until he reaches the point $P_{o}=\left(x_{o}, 0\right) \approx(2.42,0)$ then jumps into the water and swims directly to the ball.

Why is that? How strange that these two disparate strategies, one using only local information about the speed and the other using global information, should result in identical paths. Even in a simple discrete case of a taxi driver with only two choices, as Figure 2 illustrates, the globally optimal path (henceforth designated as Elvis's path) will be different from the greedy path (henceforth Salsa's path) that moves as quickly as possible toward the goal at each point. Notice as well that if Elvis and Salsa use their strategies to return from point $P_{b}$ to point $P_{a}$, then Elvis would traverse his path in reverse while Salsa would swim straight to $P_{a}$.

These observations raise the question: Under what conditions will Elvis and Salsa take identical paths and when will their paths be different? As we will see, the odds are stacked against identical paths. First of all, if the dogs started even one step into the water or on the beach, then the resulting paths would be different. Secondly, for


Figure 2. Traveling along the sides of the square from intersection A to intersection B, the greedy strategy is not optimal.
continuous speed functions, the greedy path will always be a straight line directly to the ball, and this is only optimal under very stringent conditions. Indeed, it turns out that the original problem satisfies several very special conditions that cause the two strategies to yield the same path.

## How does the location of the starting point affect the paths?

We begin by examining a generalization of the problem. Let $\left(x_{a}, y_{a}\right)$ represent the starting point and $\left(x_{b}, y_{b}\right)$ the point where the ball lands. We are assuming that the speed of the dog is dependent only on its location (not on its path), so we let $v(x, y)$ be the positive function that gives the speed of the dog at the point $(x, y)$ in all welldefined directions.

Consider the original case where $v=r$ (running speed) for $y \leq 0$ and $v=s$ (swimming speed) for $y>0$. (In this example, $v(x, 0)$ is well defined for paths with $y \leq 0$. Paths crossing the discontinuity into the water are defined piecewise.) How do Elvis's and Salsa's strategies change as their starting point varies along the $y$-axis? Notice that any path of Elvis or Salsa will be piecewise linear. Why? Their speed is constant in each medium, so they will take a straight line in each medium to get from one point to another as quickly as possible.

When the dogs start in the water $\left(y_{a}>0\right)$, they have two options: They can either swim directly toward the ball or they can swim to shore, run along the shore, and then swim out to the ball. Salsa chooses the greedy path, so she swims straight to the ball. Elvis chooses the optimal path that will depend on how far he starts from the shore. In [4], it was shown that when $y_{a}<y_{c}$ where $y_{c}$ satisfies

$$
\pm x_{b}=\frac{y_{c}+y_{b}+2 \frac{r}{s} \sqrt{y_{b} y_{c}}}{\sqrt{\left(\frac{r}{s}\right)^{2}-1}}
$$

Elvis can get to the ball fastest by swimming to the beach and running along the shore until he gets to the point $P_{o}$. When $y_{a}>y_{c}$, he is better off swimming directly to the ball. In the case $y_{a}=y_{c}$, both paths take the same amount of time.


Figure 3. Paths for four different starting points. Dashed curves represent Salsa's (greedy) path and solid curves represent Elvis's path from points of the form $P_{a}^{k}=\left(0, y_{a}^{k}\right)$ to the point $P_{b}=(5,1)$ for $r=2$ and $s=1$.

When the dogs start on the beach $\left(y_{a}<0\right)$, Salsa runs straight toward the the ball until she reaches the shore and then runs along the shore to $P_{o}$. Elvis, on the other hand, runs diagonally to the shore to a point to the right of $P_{o}$ and then swims $[\mathbf{1}, \mathbf{2}]$. Examples for both dogs are shown in Figure 3.

For a fixed landing point $P_{b}=\left(x_{b}, y_{b}\right)$, Elvis's and Salsa's paths coincide when the starting point lies along the $x$-axis, along the $y$-axis, or above the curve defined by $y_{a}=y_{c}$. The shape of this curve is shown in Figure 4. We see that the point in the original problem with $y_{a}=0$ for which the paths are identical is surrounded on both sides by starting points that give rise to different paths. Why is $y_{a}=0$ special? Obviously, this is where $v(x, y)$ is discontinuous. We shall see that Salsa's behavior changes dramatically when $v(x, y)$ is continuous.


Figure 4. Regions where the greedy and optimal paths coincide. The shaded regions, including the boundary and axes, correspond to starting points $P_{a}=\left(x_{a}, y_{a}\right)$ from which Elvis and Salsa take the same path to the destination point $P_{b}=(0,1)$ for $r=2$ and $s=1$.

Assume that $v(x, y)$ is continuous at all points $(x, y)$. Let $\mathbf{r}_{b}=\left\langle x_{b}, y_{b}\right\rangle$ represent a position vector pointing toward the tennis ball and $\mathbf{r}(t)=\langle x(t), y(t)\rangle$ the path that Salsa takes. Since $v(x, y)$ is defined everywhere and $v(x, y)=|d \mathbf{r} / d t|$,

$$
\begin{equation*}
\frac{d \mathbf{r}}{d t}=v(x(t), y(t)) \frac{\frac{d \mathbf{r}}{d t}}{\left|\frac{d \mathbf{r}}{d t}\right|} \tag{1}
\end{equation*}
$$

Salsa's distance $d_{\mathbf{r}}(t)$ from the ball along path $\mathbf{r}(t)$ is given by

$$
d_{\mathbf{r}}(t)=\left|\mathbf{r}(t)-\mathbf{r}_{b}\right|=\sqrt{\left(x(t)-x_{b}\right)^{2}+\left(y(t)-y_{b}\right)^{2}} .
$$

Taking the derivative of this with respect to $t$ we obtain

$$
\begin{equation*}
d_{\mathbf{r}}^{\prime}(t)=\frac{\left(x(t)-x_{b}\right) \frac{d x}{d t}+\left(y(t)-y_{b}\right) \frac{d y}{d t}}{\sqrt{\left(x(t)-x_{b}\right)^{2}+\left(y(t)-y_{b}\right)^{2}}}=\frac{\mathbf{r}(t)-\mathbf{r}_{b}}{\left|\mathbf{r}(t)-\mathbf{r}_{b}\right|} \cdot \frac{d \mathbf{r}}{d t} . \tag{2}
\end{equation*}
$$

Substituting (1) into (2) we have

$$
d_{\mathbf{r}}^{\prime}(t)=\left(\frac{\mathbf{r}(t)-\mathbf{r}_{b}}{\left|\mathbf{r}(t)-\mathbf{r}_{b}\right|} \cdot \frac{\frac{d \mathbf{r}}{d t}}{\left|\frac{d \mathbf{r}}{d t}\right|}\right) v(x(t), y(t)) .
$$

Since Salsa wants to decrease her distance from the ball as quickly as possible, she will choose the path $\mathbf{r}(t)$ that minimizes $d_{\mathbf{r}}^{\prime}(t)$. The dot product is minimized when the angle between $\mathbf{r}(t)-\mathbf{r}_{b}$ (which points from the tennis ball to the dog) and $d \mathbf{r} / d t$ (which points in the direction of travel) is $\pi$. So $d_{\mathbf{r}}^{\prime}(t)$ is minimized when $d \mathbf{r} / d t$ points directly toward the tennis ball. We record this as our first result.

Result 1. Wherever $v(x, y)$ is continuous, Salsa's path (the greedy path) will be a straight line toward the ball.

This somewhat surprising result relies on the fact that the speed depends only on the position of the dog and not on the path of travel, so $v(x(t), y(t))$ is constant with respect to the direction of travel.

Given this result, why did Salsa not travel along a straight line path in the original problem? Obviously, in the original problem $v(x, y)$ was not a continuous function and continuity of $v(x, y)$ was assumed for Result 1 . But where exactly was the continuity of $v(x, y)$ used? Without continuity, the above argument breaks down because we know from Darboux's theorem (an easily proved result from analysis) that it is impossible for $d \mathbf{r} / d t$ to have a jump discontinuity. Thus, $d \mathbf{r} / d t$ cannot equal the function $v(x, y)$ from the original problem [3]. For such functions, we need to analyze Salsa's path in a piecewise manner.

## When is the optimal path a straight line?

We know from the previous section that the greedy path will be a straight line toward the ball wherever $v(x, y)$ is continuous. When is this straight line path optimal? Answering this question requires the use of a technique known as calculus of variations.

Calculus of variations makes it possible to determine whether a function minimizes an integral. In our case, we want to know whether the straight line path starting at a
given point (without loss of generality, we assume this point is the origin) and ending at the tennis ball at $\left(x_{b}, y_{b}\right)$ allows Elvis to reach the tennis ball as quickly as possible.

We assume that $v(x, y)$ is not only continuous but differentiable in a region containing the straight line from the origin to the ball. Since $d t=d s / v$, where $d s$ is the differential arc length, the time it takes to travel at speed $v(x, y)$ along an arbitrary curve $C$ from $(0,0)$ to $\left(x_{b}, y_{b}\right)$ can be calculated using a line integral $T=\int_{C} d t=\int_{C}(1 / v) d s$. When we can describe this curve with a function of the form $y(x)$, this integral can be written as a functional,

$$
T(y)=\int_{0}^{x_{b}} L\left(x, y, y^{\prime}\right) d x \quad \text { where } \quad L\left(x, y, y^{\prime}\right)=\frac{\sqrt{1+\left(y^{\prime}\right)^{2}}}{v(x, y)} .
$$

If the travel time along some curve $y(x)$ is less than or equal to the travel time along any small deformation of $y(x)$ which holds the endpoints fixed (visualize plucking a guitar string), then $y(x)$ is a local minimum.

The fundamental result of calculus of variations (the proof of which can be understood by anyone with a multivariable calculus background) states that every local minimum of a functional must satisfy the Euler-Lagrange equation

$$
\begin{equation*}
\frac{\partial L}{\partial y}-\frac{d}{d x} \frac{\partial L}{\partial y^{\prime}}=0 . \tag{3}
\end{equation*}
$$

For the function $L$ defined above, we have

$$
\begin{aligned}
\frac{\partial L}{\partial y} & =-\frac{v_{y}(x, y) \sqrt{1+\left(y^{\prime}\right)^{2}}}{v(x, y)^{2}} \text { and } \\
\frac{d}{d x} \frac{\partial L}{\partial y^{\prime}} & =-\frac{y^{\prime \prime}}{\left(\sqrt{1+\left(y^{\prime}\right)^{2}}\right)^{3} v(x, y)}-\frac{y^{\prime}\left(v_{x}(x, y)+y^{\prime} v_{y}(x, y)\right)}{\sqrt{1+\left(y^{\prime}\right)^{2}} v(x, y)^{2}} v(x, y)^{2} .
\end{aligned}
$$

When does a straight line to the ball, $y(x)=\left(y_{b} / x_{b}\right) x$, satisfy this equation? (Note that if the straight line is vertical, then we can write $x=x(y)$ and take a similar approach.) For simplicity, let $m=y_{b} / x_{b}$. Thus, $y^{\prime}=m, y^{\prime \prime}=0$, and plugging this into (3) we obtain

$$
\begin{aligned}
0 & =\frac{\partial L}{\partial y}-\frac{d}{d x} \frac{\partial L}{\partial y^{\prime}} \\
& =-\frac{v_{y}(x, y) \sqrt{1+m^{2}}}{v(x, y)^{2}}+\frac{m\left(v_{x}(x, y)+m v_{y}(x, y)\right)}{\sqrt{1+m^{2}} v(x, y)^{2}} \\
& =\frac{m v_{x}(x, y)-v_{y}(x, y)}{\sqrt{1+m^{2}} v(x, y)^{2}} .
\end{aligned}
$$

This equation is satisfied when the numerator is zero. In vector form, this numerator can be written as

$$
\begin{equation*}
\nabla v(x, y) \cdot\langle m,-1\rangle=0 . \tag{4}
\end{equation*}
$$

The line $y=m x$ has a direction vector $\langle 1, m\rangle$, so $\langle m,-1\rangle$ is perpendicular to the line. Thus, the line $y=m x$ can only be a local minimum if, at each point along that line, $\nabla v(x, y)$ is parallel to the line. We state the contrapositive as our next result.

Result 2. For a differentiable function $v(x, y)$, if $\nabla v(x, y)$ is not parallel to the straight line path at every point, then the straight line path (Salsa's path) is not optimal.

Recall that $\nabla v(x, y)$ points in the direction of the greatest increase of $v(x, y)$. In other words, Elvis can increase his speed most quickly at each point by continuing in the direction of $\nabla v(x, y)$. Since the straight line path is also the shortest path, one might assume that the inverse of Result 2 is also true. Namely, if $\nabla v(x, y)$ points in the direction of the straight line path to the ball at every point along the path, then that path must be optimal. However, surprisingly, the following example shows that this is not the case.

Example. Let $v(x, y)=1+(x+1)^{2}+4 y^{2}$ and consider paths from the origin to the point $P_{b}=(4,0)$. The shortest path between these two points is a line segment along the $x$-axis shown as the dashed curve in Figure 5. Furthermore, $\nabla v(x, y)=$ $\langle 2(x+1), 8 y\rangle$, so along this line segment $\nabla v(x, y)$ is parallel to the path.

Since the time to travel along any path $y(x)$ from the origin to point $P_{b}$ satisfies

$$
T(y(x))=\int_{0}^{4} \frac{\sqrt{1+\left(\frac{d y}{d x}\right)^{2}}}{1+(x+1)^{2}+4 y^{2}} d x
$$

along the path $y(x)=0$ the travel time is $T=\arctan 5-\pi / 4 \approx 0.5880$. However, there are many paths that take less time because they take a quicker route to the regions with higher speed. For example, a path consisting of a ray and an ellipse,

$$
y(x)= \begin{cases}1.1 x & \text { if } x<1.8632 \\ \frac{1}{2} \sqrt{24-2 x-x^{2}} & \text { if } x \geq 1.8632\end{cases}
$$

shown as the solid curve in Figure 5, yields the shorter total travel time $T=0.5782$ (which was found by computing the integral numerically).

So in this example, the path that is the shortest distance and that is always in the direction of greatest change of $v(x, y)$ is not the optimal path, while another path that


Figure 5. Two different paths from $(0,0)$ to $(4,0)$ for $v(x, y)=1+(x+1)^{2}+4 y^{2}$. The dashed curve is the shortest path. However, the travel time along the solid curve is lower. The speed is indicated by the color.
enjoys neither of these properties is the optimal path. This example provides a hint for why finding the optimal path is so difficult. Unlike finding a point where a curve obtains a global minimum (which on its own can be a difficult problem), we are trying to find an entire curve that gives the minimum value. Finding a needle in a Texas-sized haystack may be easy by comparison.

## Why were the paths the same in the original problem?

Given these results, when $v(x, y)$ is continuous (which is typically the case in any natural situation), we now know exactly when the optimal and greedy paths do not coincide: Namely, Salsa always takes the greedy path directly to the ball (Result 1) and, if $\nabla v(x, y)$ is not parallel to that line, then the optimal path that Elvis takes will be different (Result 2). Thus, if $\nabla v(x, y)$ is not parallel to the line to the ball at every point, then Salsa and Elvis will not take the same path. This is a stringent condition. Why then did Elvis and Salsa travel the same path in the original configuration?

The first thing to note is that in the original problem $v(x, y)$ is not continuous, so the previous results do not apply. In order to see how the discontinuity affects their respective paths, consider the (more realistic) case where the dogs run with speed $r$ on shore, and then, once they get into the water, their speed decreases linearly (as a function of the distance from the shore) until they swim with speed $s<r$ when their feet are off the bottom. Since $v(x, y)$ is continuous, we have from Result 1 that Salsa will head directly to the ball. What will Elvis do? We can find the optimal path by solving the differential equation derived from the Euler-Lagrange equation (3). Elvis runs along the shore and then jumps into the water before getting to $P_{o}$. He then wades with speed $s<v(x, y)<r$ along a circular path until his feet are off the bottom. Finally, he swims directly toward the ball along the same path as before. Notice from Figure 6 that the narrower the wading region, the closer the resulting path is to the discontinuous solution.




Figure 6. Comparison of optimal and greedy paths. The upper panel displays the speed function $v(x, y)$ in the discontinuous case (solid) and with a wading region of width $1 / 8$ (dashed) and $1 / 2$ (dotted) when $r=2$ and $s=1$. The lower left panel displays the corresponding optimal paths, while the lower right panel displays the greedy paths (where the dashed and dotted lines overlap).

While Elvis's paths converge to the original piecewise linear solution as the wading region narrows, Salsa's path resolutely stays fixed (never say "fixed" to a dog!) as a straight line to the ball until the discontinuity occurs. Thus, we see that the discontinuity along the water's edge was necessary for the two dogs to take identical paths. How is it that the discontinuity can force a greedy strategy to become optimal?

Result 3 (Hybrid condition). Let $D$ be the distance from the starting point (the origin) to the ball, and let $d_{r}(t)$ represent the distance to the ball at time $t$ when on path $\boldsymbol{r}(t)$. If a path $\boldsymbol{r}_{o}(t)$ with $d_{r_{o}}\left(t_{f}\right)=0$ has the property that $d_{r_{o}}^{\prime}(t) \leq d_{r}^{\prime}(t)$ for a collection of paths, $\mathbf{r}(t)$, including the optimal path and for all $0 \leq t \leq t_{f}$, then $\boldsymbol{r}_{o}(t)$ is the optimal path (Elvis's path).

Proof. This result is embarrassingly easy to prove. For each path $\mathbf{r}(t), d_{\mathbf{r}}(t)$ is a function with $d_{\mathbf{r}}(0)=D$. Since $d_{\mathbf{r}}(t)$ is the integral of $d_{\mathbf{r}}^{\prime}(t)$, if $d_{\mathbf{r}_{o}}^{\prime}(t) \leq d_{\mathbf{r}}^{\prime}(t)$ holds for all $0 \leq t \leq t_{f}$, then $d_{\mathbf{r}_{o}}(t) \leq d_{\mathbf{r}}(t)$. So $d_{\mathbf{r}_{o}}\left(t_{f}\right)=0$ implies $d_{\mathbf{r}}\left(t_{f}\right) \geq 0$.

We call this the hybrid condition because it uses Salsa's strategy of considering the path moment-by-moment but achieves the globally optimal result since the greedy path is being compared with all other paths. (Similar to a coach telling a sprinter, "I don't care if you win the race; I just want you to run faster than every other runner throughout the entire race!")

As noted previously, any optimal path must consist of a straight line along the shore and a straight line through the water to the ball. So, limiting our consideration to these options, when the dogs start along the shoreline, Salsa's greedy path $\left(\mathbf{r}_{g}(t)\right)$ satisfies this condition and is therefore optimal. Why? First notice that, along this path, whenever Salsa is in the water, she is swimming with speed $s$ and is heading directly to the ball. Thus, $d_{\mathbf{r}_{g}}^{\prime}(t)$ is constant for all $t$ when Salsa is in the water. Secondly, Salsa runs along the shore as long as $d_{\mathbf{r}_{g}}^{\prime}(t)<-s$ so that the distance decreases faster by continuing along the shore. When it equals $-s$ (at point $P_{o}$ ), she jumps into the water. If Salsa would have continued running along the shore on the other side of $P_{o}$, the value of $d_{\mathbf{r}_{g}}^{\prime}(t)$ would have been greater than $-s$. Thus, $d_{\mathbf{r}_{g}}(t) \leq d_{\mathbf{r}}(t)$ for all other paths $\mathbf{r}(t)$. We conclude that the greedy path satisfies the hybrid condition and therefore must be optimal.

The authors hope the reader shares their sense of wonder at this result. When $v(x, y)$ is continuous, even if the transition between $r$ and $s$ is very steep, Salsa will always take a beeline to the ball. Why? Because her speed decreases gradually! She is lured into the water along a more direct path where her speed remains high initially but decreases rapidly as she reaches deeper waters. She would have been better off running along the shoreline where she can travel at high speeds for longer. Only when $v(x, y)$ is discontinuous along the shore does she realize that slower speeds await in the water. This leads her to deviate from the straight line and run along the shore instead. Along this path the hybrid condition is satisfied, and therefore, the greedy path becomes optimal.

## When are the paths the same for any starting point?

Are there other situations where Elvis and Salsa take identical paths? We have already seen in Figures 3 and 4 that their paths are the same when they start far enough from shore. More ambitiously, are there $v(x, y)$ for which the paths are identical for all starting points? For differentiable $v(x, y)$, we can use Results 1 and 2 to show that there are only trivial cases.

Result 4. For a differentiable speed function $v(x, y)$, the optimal path (Elvis's path) and greedy path (Salsa's path) coincide for all starting and ending points if and only if $v(x, y)$ is constant.

Proof. To see why this holds, let $(x, y)$ and $\left(x_{1}, y_{1}\right)$ be points in the $x y$-plane. First, we show that all optimal paths to a particular point $(x, y)$ are straight lines if and only if $\nabla v(x, y)=0$.

Suppose all optimal paths to $(x, y)$ are straight lines, then by Result 2 , the dot product $\left\langle x_{1}-x, y_{1}-y\right\rangle \cdot\left\langle v_{y},-v_{x}\right\rangle=0$ at each point on those paths. In particular, this must hold at the point $(x, y)$ for any $x_{1}$ and $y_{1}$. If we define $\Delta x=x_{1}-x$ and $\Delta y=y_{1}-y$, then both $\Delta x$ and $\Delta y$ are arbitrary. So, $v_{y}(x, y) \Delta x-v_{x}(x, y) \Delta y=0$ for all $\Delta x$ and all $\Delta y$. The only way for this to be true is for $v_{x}$ and $v_{y}$ to be identically 0 , meaning that $v(x, y)$ is constant.

The other direction is easy. If $v(x, y)$ is constant, the shortest path is always the optimal path. Thus, all optimal paths to $(x, y)$ must be straight lines.

Since $(x, y)$ is arbitrary, we have the result that all optimal paths to all points are straight lines if and only if $\nabla v(x, y)=0$ everywhere. We know by Result 1 that Salsa's path is always a line for differentiable $v(x, y)$, so Salsa's and Elvis's paths only coincide for all starting and ending points when $v(x, y)$ is constant.

Can Result 4 be extended to any speed function $v(x, y)$ ? If $v(x, y)$ has a discontinuity, then the two paths cannot coincide for all starting points. Why? Near the discontinuity, the problem reduces to the problem studied in Figures 3 and 4. So, for starting points near but not on the discontinuity, the greedy and optimal paths are different. It also seems likely (but as yet unproven) that this result will hold for continuous functions since the Stone-Weierstrass theorem guarantees there is a differentiable function arbitrarily close to a continuous function [3].

As a further challenge, path optimization is just one type of problem where related rate solutions can be compared with optimal solutions. How might the results of this article be applied to other situations such as maximizing overall profit versus the greedy method (literally as well as figuratively) of continually maximizing the marginal profit? More ambitiously, can these results be extended to the entire class of optimal and greedy problems? We leave those explorations, as part of the Elvis legacy, to the reader.

Summary. Elvis, the Welsh corgi, became famous when he found the quickest route down the beach and through the water to his ball. It was later discovered that Salsa, the Labrador, could achieve the same result by using a greedy approach, moving toward the ball as quickly as possible at each instant in time. We show that these paths coincide only under special conditions.

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